

Compressible perturbation of Poiseuille type flow

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Abstract. The paper examines the issue of stability of Poiseuille type flows in regime of compressible Navier-Stokes equations in a three dimensional finite pipe-like domain. We prove the existence of stationary solutions with inhomogeneous Navier slip boundary conditions admitting nontrivial inflow condition in the vicinity of constructed generic flows. Our techniques are based on an application of a modification of the Lagrangian coordinates. Thanks to such approach we are able to overcome difficulties coming from hyperbolicity of the continuity equation, constructing a maximal regularity estimate for a linearized system and applying the Banach fixed point theorem.

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1 Introduction

The mathematical description of compressible flows is important from the point of view of applications, domains such as aerodynamics and geophysics are the most natural to be mentioned here. On the other hand, complexity of the equations describing the flow delivers very interesting mathematical challenges. In spite of active research in the field, we are still far from the complete mathematical understanding of compressible flows. The only general existence results are available for weak solutions with homogeneous boundary conditions [7], [12]. As far as regular solutions are concerned, we have so far only partial results assuming either some smallness of the data, or its special structure. The problems have been investigated mainly with homogeneous boundary conditions ([4], [17]). For the overview of the state of art in the theory one can consult the monograph [18].

From the point of view of the aforementioned applications it seems very important to investigate the problems with large velocity vectors, which lead in a natural way to inhomogeneous boundary conditions. Due to the hyperbolic character of the continuity equation the density must be then prescribed on the inflow part of the boundary. Existence issues for such inflow problems are investigated in [9], [10], [19], [20], [21], [22] and [26]. The mentioned group of problems can be regarded as questions of stability of particular constant flows.

In the present article we would like to examine the issue of stability of Poiseuille type flow in pipe-like domain in compressible regime. The unperturbed flow is a solution to the compressible

Navier-Stokes system with homogeneous slip boundary conditions for given constant friction, constant density and constant external force (gravitation-like term). The Poiseuille flow is a special symmetric solution to the incompressible Navier-Stokes equations in cylindrical domains. Here it is viewed as a solution to the compressible Navier-Stokes system with constant density and constant external force, parallel to axis of the cylinder, given by the pressure. Hence the pressure, unknown in the incompressible model, is recognized as a given force. Such change of ‘observer’ looks acceptable from the mechanical point of view. Thanks to that interpretation we obtain a natural physically reasonable flow in compressible regime. The mathematical objective of this article is to establish stability of such flow under some structural assumptions limiting the magnitude of admissible perturbations.

Let us define the system. We consider steady flow of a viscous, barotropic fluid in a bounded, cylindrical domain in \mathbb{R}^3 , described by the Navier - Stokes system supplied with inhomogeneous Navier slip boundary conditions. The complete system reads

$$\begin{aligned} \rho v \cdot \nabla v - \mu \Delta v - (\mu + \nu) \nabla \operatorname{div} v + \nabla \pi(\rho) &= \rho F & \text{in } \Omega, \\ \operatorname{div}(\rho v) &= 0 & \text{in } \Omega, \\ n \cdot \mathbf{T}(v, \pi) \cdot \tau_k + f v \cdot \tau_k &= b_k, \quad k = 1, 2 & \text{on } \Gamma, \\ n \cdot v &= d & \text{on } \Gamma, \\ \rho &= \rho_{in} & \text{on } \Gamma_{in}, \end{aligned} \tag{1.1}$$

where $\Omega = [0, L] \times \Omega_0$ with $\Omega_0 \subset \mathbb{R}^2$ of class C^2 , Γ denotes the boundary of Ω (see Fig.1), v is the velocity field of the fluid, ρ its density, μ and ν are viscosity constants satisfying $\mu > 0$ and $(\nu + 2\mu) > 0$, $f \geq 0$ is the friction coefficient which may be different on different components of the boundary Γ , $\pi = \pi(\rho)$ is the pressure given as a function, at least C^1 , of the density and F is an external force. \mathbf{T} denotes the Cauchy stress tensor of the form $\mathbf{T}(v, \pi) = 2\mu \mathbf{D}v + \nu \operatorname{div} v \mathbf{Id} - \pi \mathbf{Id}$ where $\mathbf{D} = \frac{1}{2}(\nabla v + \nabla v^T)$ is the symmetric gradient. Next, n and τ_k are outer normal and tangent vectors to $\partial\Omega$. Boundary data ρ_{in}, b, d will be discussed later. The boundary Γ is naturally split into three parts:

$$\begin{aligned} \Gamma_0 &= \{x \in \partial\Omega : v(x) \cdot n(x) = 0\}, \\ \Gamma_{in} &= \{x \in \partial\Omega : v(x) \cdot n(x) < 0\}, \\ \Gamma_{out} &= \{x \in \partial\Omega : v(x) \cdot n(x) > 0\}. \end{aligned} \tag{1.2}$$

Thanks to the chosen geometry of the domain, the above decomposition is easily illustrated by the figure 1.

We shall say few words about the physical interpretation of the system (1.1), in particular about the choice of boundary conditions (1.1)_{3,4}. We would like to model a flow through a pipe. We assume that the fluid obeys Navier slip conditions on the walls of the pipe (Γ_0 component of the boundary), hence natural conditions on Γ_0 are $d \equiv 0$ and $b_k \equiv 0$. However, the mathematical requirements impose a need to prescribe the boundary conditions on Γ_{in} and Γ_{out} . From the physical viewpoint these parts are artificial, this is the area where the parameters of the velocity and density are measured. This gives us a freedom of choice of the type of boundary conditions on the inflow and outflow part, which can be fit to the mathematical approach, hence we choose inhomogeneous slip condition. Note that as the friction coefficient goes to infinity, then

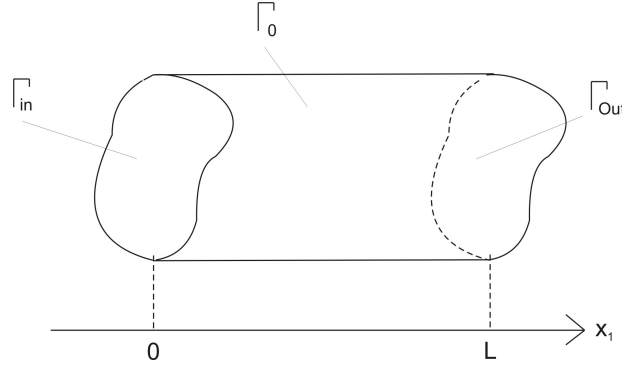


Figure 1: The domain

the relations $(1.1)_{3,4}$, at least formally, become the standard Dirichlet conditions describing the whole velocity vector at the boundary. Since the velocity does not vanish on the boundary, the hyperbolicity of the continuity equation impose a need to prescribe the density on the inflow part, which lead to the condition $(1.1)_5$. The velocity field determines the characteristics of the continuity equation and in particular the total mass $\int_{\Omega} \rho \, dx$ is determined implicitly by $(1.1)_5$.

Our goal here is to analyze a perturbation of the Poiseuille type flow

$$\bar{V} = [V^P(x_2, x_3), 0, 0], \quad (1.3)$$

where x_1 points the direction of the axis of the cylinder. It is one of the classical examples of laminar flows satisfying the incompressible Navier-Stokes equations in cylindrical domains. In the classical literature the flow is considered with homogeneous Dirichlet condition on the boundary. V^P is then found as a solution to corresponding elliptic problem with Dirichlet boundary condition on each x_1 - cut of Ω . Some explicit formulas on V^P in certain domains are well-known ([8],[11]).

In the case of slip boundary conditions that are subject of our analysis in this paper the flow (1.3) can be also found on each cut of the cylinder as a solution to elliptic problem with corresponding boundary condition (see Lemma 1 below). In certain domains it can also be expressed with explicit formulas (see [14] and the example below). Since we are interested in a general cylindrical domain, we will not have such formula but we show that the solution of the form (1.3) exists provided that Ω_0 is sufficiently regular and, under the slip boundary conditions $(1.1)_{3,4}$, V^P does not vanish on the boundary.

Lemma 1. *Let $\Omega_{\infty} = \mathbb{R} \times \Omega_0$, where $\Omega_0 \subset \mathbb{R}^2$ is smooth enough, $\mu > 0$ and $f \geq 0$. Then there exists a solution $(\bar{V}, \bar{\Pi})$, such that $\bar{V} = (V^P(x_2, x_3), 0, 0)$, to the incompressible Navier-Stokes system with slip boundary conditions:*

$$\begin{aligned} \bar{V} \cdot \nabla \bar{V} - \mu \Delta \bar{V} + \nabla \bar{\Pi} &= 0 & \text{in } \Omega_{\infty}, \\ \operatorname{div} \bar{V} &= 0 & \text{in } \Omega_{\infty}, \\ n \cdot \mathbf{T}(\bar{v}, \bar{\Pi}) \cdot \tau_k + f \bar{V} \cdot \tau_k &= 0, \quad k = 1, 2 & \text{on } \mathbb{R} \times \partial\Omega_0, \\ n \cdot \bar{V} &= 0 & \text{on } \mathbb{R} \times \partial\Omega_0. \end{aligned} \quad (1.4)$$

Moreover, there exist $\theta = \theta(f, \mu)$ and a continuous function $\bar{\omega}(f)$ such that

$$\bar{V}^{(1)} = V^P \geq \theta > 0 \quad \text{in } \bar{\Omega}_\infty. \quad (1.5)$$

and

$$\|\nabla V^P\|_{L_\infty} \leq \frac{\bar{\omega}(f)}{\mu}. \quad (1.6)$$

In addition if $(\bar{V}, \bar{\Pi})$ is the Poiseuille solution, then $(\lambda\bar{V}, \lambda\bar{\Pi})$, too.

As we already said, the pressure in the Poiseuille flow can be regarded as an external force parallel to the axis of the cylinder. Like in the classical Poiseuille flow, we assume the pressure to be a linear function of x_1 . Hence it is natural to assume the form $\bar{\pi} = \omega(f)x_1$, where $\omega(f)$ is a negative constant (the sign describes the direction of the flow). Then we see that V^P , the first coordinate of $\bar{V} = (V^P, 0, 0)$, is a solution to the elliptic problem

$$\begin{aligned} \mu \Delta V^P &= \bar{\Pi}_{x_1} = \omega(f) < 0 \quad \text{in } \Omega_0, \\ \mu \frac{\partial V^P}{\partial n} + f V^P &= 0 \quad \text{on } \partial\Omega_0. \end{aligned} \quad (1.7)$$

Now to prove Lemma 1 it is enough to apply the maximum principle to the system (1.7). We show the proof in the Appendix, at this stage we should have a closer look at the dependence $\omega(f)$. This dependence can be justified considering the compatibility condition for the system (1.7), that reads $\int_{\Omega_0} \omega dx = f \int_{\partial\Omega_0} V^P d\sigma$. As we will see, the dependence $\omega(f)$ determines the assumptions we will have to make on the viscosity.

Note that for $\omega = 0$ the only solution is $V^P = 0$, provided $f \neq 0$. For $f = 0$ (perfect slip) we should obtain a constant flow, hence we put $\omega = 0$ for $f = 0$. The linear structure of (1.7) makes $\omega(f)$ continuous. Thus, we conclude $\omega(f) \rightarrow 0$ for $f \rightarrow 0$, and hence by (1.6)

$$\|\nabla \bar{V}\|_{L_\infty} \rightarrow 0 \quad \text{for } f \rightarrow 0. \quad (1.8)$$

This is an important conclusion since in order to show the energy estimate we have to control $\nabla \bar{V}$ with the viscosity, and so we allow the viscosity to be low provided that the friction on Γ_0 is small, what is a realistic assumption (see also the remarks after the formulation of Theorem 1). Let us illustrate the dependence $\omega = \omega(f)$ with the following example.

Example. Take $\Omega_0 = B(0, 1) \subset \mathbb{R}^2$ and $\mu = 1$. Then, due to axial symmetry of the domain, it is natural to look for $V^P = v^f(r)$ where $r = \sqrt{x_2^2 + x_3^2}$. The boundary condition (1.7)₂ then reads $v_r^f + f v^f|_{r=1} = 0$. We require that $\Delta v = v_{rr} + \frac{1}{r} v_r$ depend only on f . Moreover, we expect to obtain a constant flow for $f = 0$ (perfect slip) and classical Poiseuille profile for $f = \infty$. The above considerations lead to the family of solutions

$$v^f(r) = TF \left[\frac{f+2}{f+k_f} - \frac{f}{f+k_f} r^2 \right],$$

where $k_f = \frac{(\pi-2)f+4\pi}{2}$ and TF is the flux of the flow v^f through Ω_0 .

For a perfect slip case $f = 0$ we obtain a constant flow $v^0 = \frac{2TF}{k_f}$ and for a no-slip case $f = \infty$ we get a classical Poiseuille profile $v^\infty = TF(1 - r^2)$. On the boundary we have $\theta(f) = v^f(1) = \frac{2}{f+k_f}$ what is a strictly positive constant. Finally,

$$\omega(f) = \Delta v^f = -\frac{4f}{f+k_f} \quad \text{and} \quad \nabla v^f \sim \frac{f}{f+k_f}.$$

In particular $\omega(f) < 0$ for $f > 0$ and $\omega(0) = 0$.

Before we formulate our main result, we need one observation concerning the boundary conditions. Note that, since V^P is found on every x_1 - cut of Ω , we can impose the boundary conditions (1.4)_{3,4} only on Γ_0 . On the other hand, in order to define small perturbations as a solution to (1.1) we have to measure the distance (in appropriate norms) between the solution to (1.1) and the Poiseuille flow \bar{V} . Hence we need to consider the traces of the quantities from the boundary conditions of (1.1) with the function \bar{V} instead of v . Since our analysis acts on a finite cylinder we define these traces at the bottoms $\Gamma_{in} \cup \Gamma_{out}$:

$$\bar{b}_k|_{\Gamma_{in} \cup \Gamma_{out}} = \text{tr}_{\Gamma_{in} \cup \Gamma_{out}}[n \cdot \mathbf{T}(\bar{V}, \bar{\Pi}) \cdot \tau_k + f\bar{V} \cdot \tau_k], \quad \bar{d}|_{\Gamma_{in}} = -\text{tr}_{\Gamma_{in}} V^P, \quad \bar{d}|_{\Gamma_{out}} = \text{tr}_{\Gamma_{out}} V^P,$$

where tr denotes the trace operator. By (1.4),

$$n \cdot \mathbf{T}(\bar{V}, \bar{\Pi}) \cdot \tau_k + f\bar{V} \cdot \tau_k = 0 \quad \text{and} \quad n \cdot \bar{V} = 0 \quad \text{at } \Gamma_0.$$

Hence the construction of Poiseuille flow determines ($k = 1, 2$):

$$\bar{b}_k = \bar{d} = 0 \quad \text{on } \Gamma_0. \tag{1.9}$$

The Poiseuille flow $(\bar{V}, \bar{\Pi})$ has constant density $\bar{\rho}$, we set $\bar{\rho} = 1$. Then in the chosen setting $(\bar{V}, \bar{\Pi})$ fulfills the following system

$$\begin{aligned} \bar{\rho}\bar{V} \cdot \nabla \bar{V} - \mu\Delta \bar{V} - (\mu + \nu)\nabla \text{div } \bar{V} + \nabla \bar{\Pi}(\bar{\rho}) &= -\bar{\rho}\omega(f)\hat{e}_1 & \text{in } \Omega, \\ \text{div}(\bar{\rho}\bar{V}) &= 0 & \text{in } \Omega, \\ n \cdot \mathbf{T}(\bar{V}, \bar{\Pi}) \cdot \tau_k + f\bar{V} \cdot \tau_k &= \bar{b}_k, \quad k = 1, 2 & \text{on } \Gamma, \\ n \cdot \bar{V} &= \bar{d} & \text{on } \Gamma. \end{aligned} \tag{1.10}$$

We keep in mind that $\nabla \bar{\Pi} = \omega(f)\hat{e}_1$ and (1.9).

We are now in a position to formulate our main result. To this end it is convenient to define the quantity which measures the distance of the data from the Poiseuille flow:

$$D_0 = \|F + \omega(f)\hat{e}_1\|_{L_p(\Omega)} + \|d - \bar{d}\|_{W_p^{2-1/p}(\Gamma)} + \|b_k - \bar{b}_k\|_{W_p^{1-1/p}(\Gamma)} + \|\rho_{in} - 1\|_{W_p^1(\Gamma_{in})}. \tag{1.11}$$

The main result of the paper reads

Theorem 1. *Assume that the boundary data of (1.1) is close to the Poiseuille flow $(\bar{V}, \bar{\Pi})$, more precisely, let D_0 defined above be small enough. Assume that the friction f is large enough on Γ_{in} and $p > 3$. Assume further that there exists a constant $\kappa > 0$ such that*

$$\mu > \kappa \min\{f|_{\Gamma_0}, 1\}. \tag{1.12}$$

Then there exists a solution $(v, \rho) \in W_p^2(\Omega) \times W_p^1(\Omega)$ to the system (1.1) such that

$$\|v - \bar{V}\|_{W_p^2(\Omega)} + \|\rho - \bar{\rho}\|_{W_p^1(\Omega)} \leq C(D_0). \quad (1.13)$$

This solution is unique in the class of small perturbations of $(\bar{V}, \bar{\rho})$.

Let us make some remarks concerning our main result. The condition on the viscosity (1.12) seems to be a serious constraint, but as we will see from the proofs we just need it to control the gradient of the Poiseuille flow, what yields this assumption natural (see also the remark in the proof of Lemma 6). We recall that $\nabla \bar{V}$ depends on the friction f on Γ_0 , and in particular (1.8) holds. It follows that for small values of friction on Γ_0 it is enough to assume that the viscosity is large enough, but only compared to the friction. This assumption is reflected in the condition (1.12). Theorem 1 admits the case of perfect slip $f|_{\Gamma_0} = 0$, and in such case (1.12) reduces to $\mu > 0$, so no lower bound on the viscosity is required. In this case there is no bound on the size of \bar{V} . However in this case \bar{V} would be a constant flow. We shall recall that the friction at Γ_{in} is chosen independently to f at Γ_0 . From the point of view of modelling, the data at Γ_{in} is given, hence it is important to focus the attention at Γ_0 . The assumption $p > 3$ is required for the imbedding $W_p^1 \subset L_\infty$ [1], it is required to control the pointwise boundedness of ∇v and the density.

Let us explain the main idea of the proof. We will follow an idea of Lagrangian type coordinates [5], [13], [16], [24]. The continuity equation is of hyperbolic type and contains a term $u \cdot \nabla w$ (where u and w are perturbations to the velocity and density introduced in the next section), which makes serious troubles for the issues of existence in case of inhomogeneous boundary conditions, see [9], [21], [18], [19], [20]. Here we overcome this obstacle by changing the system of coordinates in such a way that this term disappears (2.7). We obtain a more complex system but with structure suitable for an application of the Banach fixed point theorem. On the other hand our solutions are regular enough, thus we are able to go back to the original system keeping the well posedness of the original model. Our approach works since we are equipped with the maximal regularity estimate for a linearization of the equations in the Lagrangian coordinates – Theorem 2. This tool gives a complete control of the regularity of solutions.

The rest of the paper is organized as follows. In Section 2 we introduce the perturbations as unknown variables obtaining the system (2.5). Next we introduce the Lagrangian-type coordinates that lead to the system (2.11) and we derive the necessary estimates for the Lagrangian transformation. In Section 3 we deal with the linearization of (2.11). For the linear system we show the estimate in $W_p^2(\Omega) \times W_p^1(\Omega)$. It is given by Theorem 2. The first step is the energy estimate (3.3). Then we consider the vorticity of the velocity and the Helmholtz decomposition to reduce the continuity equation to a sort of transport equation (3.22) that enables us to find the bound on $\|w\|_{W_p^1(\Omega)}$. This result together with the properties of the Lamé system lets us conclude Theorem 2. In the second part of this section we apply the estimates to solve the linear system and hence show that T given by (2.27) is well defined. In Section 4 we show the contraction principle for T . To this end we consider the system for the difference of two solutions and write it in a form (4.1) which has a structure of (3.1). The contraction results from the estimate (3.30) and bounds on the norms on the r.h.s. of the system for the difference. At the end of Section

4 we apply the Banach fixed point theorem to solve the system (2.11) and conclude the proof of Theorem 1.

Let us finish this introductory part with some remarks concerning notation. By C we shall denote a constant that is controlled, but not necessarily small. E shall denote a constant that can be arbitrarily small provided the data is small enough. Sometimes we will write $E(\cdot)$ to underline that we need the smallness of certain quantity. The functional spaces on Ω will be denoted without the symbol of the set, for example we will write W_p^k instead of $W_p^k(\Omega)$ for standard Sobolev spaces of functions intergable with the p -th power with derivatives up to order k , $W_p^{1-1/p}(\partial\Omega)$ denotes the Slobodeckij spaces, defining regularity of traces from $W_p^1(\Omega)$, [1]. Finally, we will need to consider the density in the space $L_\infty(0, L; L_2(\Omega_0))$. For simplicity we denote it as $L_\infty(L_2)$. We do not use different notation for scalar and vector valued functions, while matrix valued functions are written in bolded font. The coordinates of a vector are denoted by (\cdot) , i.e. $u = (u^{(1)}, u^{(2)}, u^{(3)})$.

2 Preliminaries

In this section we introduce perturbations of the Poiseuille flow $(\bar{V}, \bar{\rho})$ as unknown variables, what leads to the system (2.5). Then we introduce a change of variables that straightens the characteristics of the continuity equation. We obtain the system (2.11). The simplified form of the continuity equation in this Lagrangian framework makes it possible to apply the Banach fixed point theorem to the system (2.11).

2.1 Reformulation of the problem

We come back to the main system (1.1). Since we are interested in solutions that are small perturbations of $(\bar{V}, \bar{\rho})$, it is convenient to consider the perturbations as unknown functions. For technical reasons it is better to have $u \cdot n = 0$ on the boundary. Hence we start introducing $u_0 \in W_p^2(\Omega)$ such that $u_0 \cdot n|_\Gamma = d - \bar{d}$ (recall that $\bar{d} = \bar{V} \cdot n$). It can be found as $u_0 = \nabla \phi$ where ϕ solves a Neumann problem. We assume that $\|d - \bar{V} \cdot n\|_{W_p^{1-1/p}(\Gamma)}$ is small enough for

$$V^P + u_0^{(1)}|_{\bar{\Omega}} \geq \theta_1 \quad (2.1)$$

to hold for some $\theta_1 > 0$. In fact this is not really a restriction as we consider small perturbations of $(\bar{V}, \bar{\rho})$ and in Lemma 1 we have shown that V^P is separated from zero. Now we take

$$u = v - \bar{V} - u_0. \quad (2.2)$$

In particular we want our perturbed flow v to have the first component also separated from zero. This is quite natural constraint if we consider small perturbations of \bar{V} . With the above definition of u this constraint reads

$$V^P + u^{(1)} + u_0^{(1)} \geq \theta_2 > 0. \quad (2.3)$$

Next we introduce the perturbation of the density (recall that $\bar{\rho} \equiv 1$):

$$w = \rho - 1. \quad (2.4)$$

Substituting (2.2) and (2.4) to (1.1) and (1.10) we arrive at

$$\begin{aligned}
u \cdot \nabla \bar{V} + V^P \partial_{x_1} u - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + \gamma \nabla w &= F(u, w), \\
V^P \partial_{x_1} w + \operatorname{div} u + (u + u_0) \cdot \nabla w &= G(u, w), \\
n \cdot 2\mu \mathbf{D}(u) \cdot \tau_k + f(u \cdot \tau_k)|_\Gamma &= B_k, \\
n \cdot u|_\Gamma = 0, \quad w|_{\Gamma_{in}} &= w_{in},
\end{aligned} \tag{2.5}$$

where $\gamma = \pi'(1)$ and

$$\begin{aligned}
F(u, w) &= (u + u_0) \cdot \nabla(u + u_0) - u_0 \cdot \nabla \bar{V} - V^P \partial_{x_1} u_0 - [\pi'(w + 1) - \pi'(1)] \nabla w \\
&\quad - w \omega(f) \hat{e}_1 + (w + 1)(F + \omega(f) \hat{e}_1) - w(u + u_0 + \bar{V}) \cdot \nabla(u + u_0 + \bar{V}), \\
G(u, w) &= -w \operatorname{div} u - (w + 1) \operatorname{div} u_0, \\
B_k &= b_k - n \cdot 2\mu \mathbf{D}(\bar{V} + u_0) \cdot \tau_k - f(\bar{V} + u_0) \cdot \tau_k.
\end{aligned}$$

From now on we focus on the system (2.5). Notice that $\bar{V} \cdot \nabla \bar{V} = 0$, hence the term $-w(u + u_0 + \bar{V}) \cdot \nabla(u + u_0 + \bar{V})$ is a higher order term and so the form of F and G implies immediately the following lemma.

Lemma 2. *Let $F(u, w)$ and $G(u, w)$ be given as above, then*

$$\|F(u, w)\|_{L_p} + \|G(u, w)\|_{W_p^1} \leq C[(\|u\|_{W_p^2} + \|w\|_{W_p^1})^2 + E(\|u\|_{W_p^2} + \|w\|_{W_p^1}) + D_0], \tag{2.6}$$

where E denotes a small, compared to μ , positive constant.

2.2 Change of variables

With our smallness assumptions it is quite natural to solve (2.5) with a fixed point argument. However, a direct application of this method fails because of the nonlinear term $u \cdot \nabla w$ in the hyperbolic continuity equation. The idea to overcome this problem is to introduce a change of variables such that this awkward term vanishes. We look for the appropriate transformation as $x = \psi_{u+u_0}(z)$ satisfying the identity

$$V^P \partial_{z_1} = V^P \partial_{x_1} + (u + u_0) \cdot \nabla_x. \tag{2.7}$$

In the following lemma we construct the mapping $\psi_{\bar{u}}$ for arbitrary function \bar{u} small in W_p^2 with vanishing normal component on the boundary Γ_0 .

Lemma 3. *Let $\|\bar{u}\|_{W_p^2}$ be small enough and $\bar{u} \cdot n|_{\Gamma_0} = 0$. Then there exists a diffeomorphism $x = \psi_{\bar{u}}(z)$ defined on Ω such that $\Omega = \psi_{\bar{u}}(\Omega)$ and (2.7) holds with $u + u_0 = \bar{u}$.*

Proof. A key point in the proof is the fact that $V^P \geq c > 0$. In particular we are able to divide (2.7) by V^P obtaining

$$\partial_{z_1} = \partial_{x_1} + \tilde{u} \cdot \nabla_x,$$

where $\tilde{u} = \frac{\bar{u}}{V^P}$. Since $\bar{u}, V^P \in W_p^2$ we have $\tilde{u} \in W_p^2$ and, since we are interested in small perturbations we can assume that

$$\|\tilde{u}\|_{W_p^2} \ll 1. \quad (2.8)$$

Now we can follow the proof from [20] and look for $\psi(z_1, z_2, z_3) = \psi_{z_2, z_3}(z_1)$, where for each $(z_2, z_3) \in \Gamma_0$ the function ψ_{z_2, z_3} is a solution to

$$\begin{cases} \partial_s \psi_{z_2, z_3}^{(1)} = 1, & \partial_s \psi_{z_2, z_3}^{(2)} = \tilde{u}^2(\psi_{z_2, z_3}), & \partial_s \psi_{z_2, z_3}^{(3)} = \tilde{u}^{(3)}(\psi_{z_2, z_3}), \\ \psi_{z_2, z_3}(0) = (0, z_2, z_3). \end{cases} \quad (2.9)$$

Due to (2.8) we solve (2.9) for $(z_2, z_3) \in \Gamma_{in}$ following [20] and show that there exists a set $\Omega_{\bar{u}}$ such that $\psi(\Omega_{\bar{u}}) \rightarrow \Omega$ is a diffeomorphism. It remains to show that $\Omega_{\bar{u}} = \Omega$. To this end we examine the derivatives of ψ . We have $D\psi = \text{Id} + \mathbf{E}$ where

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{u}^2(\psi(z)) & \partial_{z_2} \int_0^{z_1} \tilde{u}^2(\psi(s, \bar{z})) ds & \partial_{z_3} \int_0^{z_1} \tilde{u}^2(\psi(s, \bar{z})) ds \\ \tilde{u}^{(3)}(\psi(z)) & \partial_{z_2} \int_0^{z_1} \tilde{u}^{(3)}(\psi(s, \bar{z})) ds & \partial_{z_3} \int_0^{z_1} \tilde{u}^{(3)}(\psi(s, \bar{z})) ds \end{bmatrix} \quad (2.10)$$

and $\bar{z} := (z_2, z_3)$. The first row of \mathbf{E} reduces to 0 since $\mathbf{E}_{1,i} = \partial_{z_i} \int_0^{z_1} ds = 0$ for $i = 2, 3$. Hence

$$D\psi([1, 0, 0]) = [1, \tilde{u}^2(\psi(z)), \tilde{u}^{(3)}(\psi(z))].$$

Now take $x_n \rightarrow x_0 \in \Gamma_0$ and $z_n = \phi(x_n)$. Then we have

$$\lim_{n \rightarrow \infty} D\psi(z_n)([1, 0, 0]) = [1, \tilde{u}^2(x_0), \tilde{u}^{(3)}(x_0)].$$

The latter is parallel to Γ_0 since $\tilde{u} \cdot n|_{\Gamma_0} = 0$, hence we have $\phi(\Gamma_0) = \Gamma_0$ (precisely we say it in a sense of tangent spaces since ϕ is defined only on Ω). To examine the behavior of tangent vectors on Γ_{out} notice that

$$D\psi(z)([0, \tau_1, \tau_2]) = ([0, \tilde{\tau}_1(\psi(z)), \tilde{\tau}_2(\psi(z))]),$$

where $\tilde{\tau}_i$ are given by appropriate entries of $D\psi$, the important fact is that the first coordinate vanishes. Hence if we take $x_n \rightarrow x_0$ and $z_n = \phi(x_n)$, this time with $x_0 \in \Gamma_{out}$, then

$$\lim_{n \rightarrow \infty} D\psi(z_n)([0, \tau_1, \tau_2]) = [0, \tilde{\tau}_1(x_0), \tilde{\tau}_2(x_0)],$$

what is parallel to Γ_{out} . It shows that $\phi(\Gamma_{out}) = \Gamma_{out}$. Since $\phi(\Gamma_0) = \Gamma_0$ by the definition of ϕ , we conclude that $\Omega_{\bar{u}} = \Omega$ and complete the proof. \square

Now we proceed with transformation of the system (2.5). As a vector field \bar{u} satisfying the assumptions of the Lemma we take $u + u_0$ where u is the solution of (2.5). So far we don't know if this solution exists, our goal is to show its existence. Hence our approach can be regarded as working in a kind of Lagrangian coordinates [16]; assuming that the solution exists we rewrite the system in the new variables ψ_{u+u_0} induced by u through (2.7). Then in the new coordinates we hope to be able to apply a fixed point method to show the existence of a solution. Since (u, w) are perturbations that are assumed to be small, we can assume that $\|u\|_{W_p^2}$ is small enough that

the assumptions of Lemma 3 are satisfied. Hence the solution gives a well defined transformation $x = \psi_{u+u_0}(z)$, that we denote for simplicity by ψ . If we show in addition its uniqueness in a class of small perturbations, then denoting $\phi = \psi^{-1}$ we have $\psi(\Omega_u) = \Omega$ and we come back to the original coordinates where our solution solves (2.5). Rewriting the system (2.5) in coordinates z yields

$$\begin{aligned} u \cdot \nabla_z \bar{V} + (V^P \circ \psi_u) \partial_{z_1} u - \mu \Delta_z u - (\mu + \nu) \nabla_z \operatorname{div}_z u + \gamma \nabla_z w &= \tilde{F}(u, w), \\ (V^P \circ \psi_u) \partial_{z_1} w + \operatorname{div}_z u &= \tilde{G}(u, w), \\ n \cdot 2\mu \mathbf{D}_z(u) \cdot \tau_k + f(u \cdot \tau_k)|_\Gamma &= B_k - n \cdot 2\mu R(u, \mathbf{D}) \cdot \tau_k, \\ n \cdot u|_\Gamma &= 0, \quad w|_{\Gamma_{in}} = w_{in}. \end{aligned} \tag{2.11}$$

The functions \tilde{F} and \tilde{G} involve commutators from the change of variables. More precisely,

$$\tilde{F}(u, w) = F(u, w) - u \cdot R(\bar{V}, \nabla) - V^P R(u, \partial_{x_1}) + \mu R(u, \Delta) + (\mu + \nu) R(u, \nabla \operatorname{div}) - \gamma R(w, \nabla) \tag{2.12}$$

and

$$\tilde{G}(u, w) = G(u, w) - R(u, \operatorname{div}). \tag{2.13}$$

Here the first variable in the commutator $R(\cdot, \cdot)$ denotes a function and the second is a differential operator. For example, $R(w, \nabla) := \nabla_x w - \nabla_z w$ and its i -th coordinate reads

$$R^{(i)}(w, \nabla) = [\partial_{z_i} w(\phi_{x_i}^{(i)} - 1) + \sum_{i \neq j} w_{z_j} \phi_{x_i}^{(j)}].$$

We shall not give here precise formulas for the other commutators. Instead, we are now ready to give some heuristic arguments that will show what regularity we expect from the change of variables ϕ . To this end note that the commutators of the operators of order k depends on the derivatives of ϕ up to order k . More precisely, commutators of order one contain only components of the form

$$\nabla_z u \cdot \nabla_x \phi,$$

while second-order commutators contain the terms

$$\nabla_z^2 u \cdot (\nabla_x \phi)^2, \quad \nabla_z u \cdot \nabla_x^2 \phi$$

and the terms of lower order. Hence in order to find the estimates on $\|\tilde{F}(u, w)\|_{L_p}$ and $\|\tilde{G}(u, w)\|_{W_p^1}$ we need

$$\nabla_x \phi \in L_\infty, \quad \nabla_x^2 \phi \in L_p,$$

what will be satisfied provided that $\phi \in W_p^2$ due to the imbedding $W_p^1 \in L_\infty$ (recall that $p > 3$). We should also note that, for simplicity of notation, the functions $F(u, w)$ and $G(u, w)$ in (2.12) and (2.13) denote exactly the same quantities as before. Hence we should keep in mind that they contain differential operators and so now they also contain some commutators that we will have to control to repeat the estimate (2.6).

In order to show such estimate we should have a closer look at the derivatives of ϕ . With our construction of $\psi = \phi^{-1}$, it is easier to consider first the derivatives of ψ that are given by (2.10)

with $\tilde{u} = \frac{u+u_0}{V^P}$ where u is the solution to (2.5), not to be confused with $\tilde{u} = \frac{\bar{u}}{V^P}$ from the proof of Lemma 3 where \bar{u} was an arbitrary function. In order to find the bounds on the commutators we will need the following smallness results for our change of variables:

Lemma 4. *We have*

$$\sum_i \left\| \frac{\partial \psi^{(i)}}{\partial z_i} - 1 \right\|_{W_p^1} + \sum_{i \neq j} \left\| \frac{\partial \psi^{(i)}}{\partial z_j} \right\|_{W_p^1} \leq E, \quad (2.14)$$

$$\|D_z^2 \psi\|_{L_p} \leq E, \quad (2.15)$$

$$\sum_i \left\| \frac{\partial \phi^{(i)}}{\partial x_i} - 1 \right\|_{W_p^1} + \sum_{i \neq j} \left\| \frac{\partial \phi^{(i)}}{\partial x_j} \right\|_{W_p^1} \leq E, \quad (2.16)$$

$$\|D_x^2 \phi\|_{L_p} \leq E, \quad (2.17)$$

where E is sufficiently small number comparing to μ , depending on norms of perturbations measured by D_0 defined in (1.11).

Proof. The core of the proof is in the imbedding $W_p^1 \subset L_\infty$. We start with (2.14). We estimate L_p norm of the entries of \mathbf{E} (2.10). These results quite directly from the form of \mathbf{E} , but needs certain attention as \mathbf{E} depends on ψ implicitly. The entries of \mathbf{E} without integrals will be small provided that ψ is bounded what obviously holds true. For the entries involving integrals we change the order of integration and derivative obtaining

$$\partial_{z_i} \psi^{(j)} = \partial_{z_i} \int_0^{z_1} \tilde{u}(\psi(s, \bar{z})) ds = \int_0^{z_1} [\nabla_x \tilde{u}^{(j)}(\psi(s, \bar{z})) \cdot \partial_{z_i} \psi(s, \bar{z})] ds.$$

By Jensen inequality we have

$$\int_0^{z_1} |\partial_{x_k} \tilde{u}^{(j)}(\psi(s, \bar{z}))|^p |\partial_{z_i} \psi^{(k)}(s, \bar{z})|^p ds \leq |z_1|^{p-1} \|\nabla_x \tilde{u}\|_{L_\infty}^p \int_0^{z_1} |\partial_{z_i} \psi^{(k)}(s, \bar{z})|^p ds.$$

Integrating the last inequality over Ω we get

$$\|E_{ij}\|_{L_p} \leq C(1 + \sum_{k,l} \|E_{kl}\|_{L_p}) \|\nabla_x \tilde{u}\|_{L_\infty}. \quad (2.18)$$

The smallness of \tilde{u} in W_p^2 and the imbedding $W_p^1 \subset L_\infty$ gives (2.14).

To show (2.15) we differentiate the entries of \mathbf{E} , let us focus on entries with integrals. We have (we omit the sum over k):

$$\begin{aligned} \partial_{z_l} \partial_{z_i} \psi^{(j)} &= \partial_{z_l} \int_0^{z_1} [\partial_{x_k} \tilde{u}^{(j)}(\psi(s, \bar{z})) \partial_{z_i} \psi^{(k)}(s, \bar{z})] ds = \\ &= \int_0^{z_1} \partial_{z_l} [\partial_{x_k} \tilde{u}^{(j)}(\psi(s, \bar{z}))] \partial_{z_i} \psi^{(k)}(s, \bar{z}) ds + \int_0^{z_1} \partial_{x_k} \tilde{u}^{(j)} \psi(s, \bar{z}) \partial_{z_l} \partial_{z_k} \psi^{(k)}(s, \bar{z}) ds =: I_1 + I_2. \end{aligned}$$

Again by Jensen inequality,

$$|I_1|^p = \left| \int_0^{z_1} \partial_{x_m} \partial_{x_k} \tilde{u}^{(j)}(\psi(s, \bar{z})) \partial_{z_l} \psi^{(m)}(s, \bar{z}) \partial_{z_i} \psi^{(k)}(s, \bar{z}) ds \right|^p \leq \|\nabla_z \psi\|_{L_\infty}^2 p |z_1|^{p-1} \int_0^{z_1} |\partial_{x_m} \partial_{x_k} \tilde{u}^{(j)}|^p ds \quad (2.19)$$

and

$$|I_2|^p \leq \|\nabla_x \tilde{u}\|_{L_\infty}^p \int_0^{z_1} |\partial_{z_l} \partial_{z_k} \psi^{(k)}(s, \bar{z})|^p ds.$$

Integrating the above inequalities over Ω we arrive at

$$\|\nabla^2 \psi\|_{L_p} \leq C(\Omega) [\|\nabla_x \tilde{u}\|_{L_\infty} \|\nabla^2 \psi\|_{L_p} + \|\nabla_x^2 \tilde{u}\|_{L_p}], \quad (2.20)$$

where the term $\|\nabla_z \psi\|_{L_\infty}^2$ from (2.19) has been put into the constant. Like in the previous estimate, the imbedding $W_p^1 \subset L_\infty$ and the smallness of \tilde{u} in W_p^2 yield (2.15).

To show (2.16) note that the smallness of \mathbf{E} given by (2.14) combined with the imbedding $W_p^1 \subset L_\infty$ implies that $\det D\psi \geq c > 0$, hence $D\psi$ is invertible and we have

$$D\phi = D\psi^{-1} = \mathbf{Id} + \tilde{\mathbf{E}},$$

where the elements of $\tilde{\mathbf{E}}$ can be explicitly computed in terms of \mathbf{E} . The smallness of \tilde{u} together with the fact that W_p^1 is an algebra implies smallness in L_p of the entries of $\tilde{\mathbf{E}}$, which gives (2.16). Finally (2.17) is obtained by taking derivatives of $D\phi$ similarly to (2.15). \square

Now we are ready to show the basic estimate on the r.h.s. of (2.11):

Lemma 5. *Let \tilde{F} and \tilde{G} be defined by (2.12) and (2.13). Then we have*

$$\|\tilde{F}(u, w)\|_{L_p} + \|\tilde{G}(u, w)\|_{W_p^1} \leq C[(\|u\|_{W_p^2} + \|w\|_{W_p^1})^2 + D_0] + E(\|u\|_{W_p^2} + \|w\|_{W_p^1}). \quad (2.21)$$

Proof. The bound on $\|F(u, w)\|_{L_p} + \|G(u, w)\|_{W_p^1}$ results from (2.6) in Lemma 2 (there are also some commutators since F and G involve differential operators, but these can be estimated as follows). We briefly justify the bounds on the commutators in \tilde{F} . To start with, the first order commutators contain the given function \tilde{V} , the functions u and w and the derivatives of ϕ , but only of the form

$$\phi_{x_j}^{(i)}, \quad i \neq j \quad \text{and} \quad \phi_{x_i}^{(i)} - 1. \quad (2.22)$$

Hence applying Lemma 4 we get

$$\|u \cdot R(\tilde{V}, \nabla)\|_{L_p} + \|V^P R(u, \partial_{x_1})\|_{L_p} + \|R(w, \nabla)\|_{L_p} \leq E(\|u\|_{W_p^2} + \|w\|_{W_p^1}). \quad (2.23)$$

The second order commutators contain the second order derivatives of ϕ and the first order derivatives only of the form (2.22). Hence the application of Lemma (4) yields

$$\|R(u, \Delta)\|_{L_p} + \|R(u, \nabla \operatorname{div})\|_{L_p} \leq E\|u\|_{W_p^2} \quad (2.24)$$

and we conclude the bound on $\|\tilde{F}\|_{L_p}$. In order to estimate $\|\tilde{G}(u, w)\|_{W_p^1}$ we differentiate the commutator

$$R(u, \operatorname{div}) = \sum u_{z_i}^{(i)} (\phi_{x_i}^{(i)} - 1) + \sum_{i \neq j} u_{z_j}^{(i)} \phi_{x_i}^{(j)}(x),$$

what yields

$$\begin{aligned}\partial_{z_k} R(u, \text{div}) &= \sum_i \left[u_{z_i z_k}^{(i)} (\phi_{x_i}^{(i)} - 1) + u_{z_i}^{(i)} \sum_j \phi_{x_i x_j}^{(i)} \psi_{z_k}^{(j)} \right] \\ &+ \sum_{i \neq j} \left[u_{z_j z_k}^{(i)} \phi_{x_i}^{(j)}(x) + u_{z_j}^{(i)} \sum_l \phi_{x_i x_l}^{(j)} \psi_{z_k}^{(l)} \right].\end{aligned}$$

Applying again Lemma 4 we get

$$\|R(u, \text{div})\|_{W_p^1} \leq E \|u\|_{W_p^2} \quad (2.25)$$

and the proof is complete. \square

From now on we focus on the system (2.11) instead of (2.5). It is of crucial importance for us that we can solve (2.11) in the domain Ω , what results from our choice of the transformation ψ .

Now we are in a position to define the operator

$$T : W_p^2(\Omega) \times W_p^1(\Omega) \rightarrow W_p^2(\Omega) \times W_p^1(\Omega), \quad (2.26)$$

to which we want to apply the Banach fixed point theorem in order to solve the system (2.11). Namely, we set $(u, w) = T(\bar{u}, \bar{w})$ if

$$\begin{aligned}u \cdot \nabla_z \bar{V} + (V^P \circ \psi_{\bar{u}+u_0}) \partial_{z_1} u - \mu \Delta_z u - (\mu + \nu) \nabla_z \text{div}_z u + \gamma \nabla_z w &= F(\bar{u}, \bar{w}), \\ ((V^P \circ \psi_{\bar{u}+u_0})) \partial_{z_1} w + \text{div}_z u &= G(\bar{u}, \bar{w}), \\ n \cdot 2\mu \mathbf{D}_z(u) \cdot \tau_k + f(u \cdot \tau_k)|_\Gamma &= B_k, \\ n \cdot u|_\Gamma &= 0, \quad w|_{\Gamma_{in}} = w_{in}.\end{aligned} \quad (2.27)$$

The point is that the term $\partial_{x_1} w + u \cdot \nabla w$ is replaced by $((V^P \circ \psi_{\bar{u}}) + u^{(1)}) \partial_{z_1} w$, and for this term we find a bound in W_p^1 , what is necessary to show the contraction property of T . From now on for simplicity we will write $\psi_{\bar{u}}$ instead of $\psi_{\bar{u}+u_0}$.

3 A priori bounds and solution of the linear system

In this section we deal with the linear system:

$$\begin{aligned}u \cdot \nabla_z \bar{V} + (V^P \circ \psi_{\bar{u}}) \partial_{z_1} u - \mu \Delta_z u - (\mu + \nu) \nabla_z \text{div}_z u + \gamma \nabla_z w &= F, \\ ((V^P \circ \psi_{\bar{u}})) \partial_{z_1} w + \text{div}_z u &= G, \\ n \cdot 2\mu \mathbf{D}_z(u) \cdot \tau_k + f(u \cdot \tau_k)|_\Gamma &= B_k, \\ n \cdot u|_\Gamma &= 0, \quad w|_{\Gamma_{in}} = w_{in},\end{aligned} \quad (3.1)$$

with given functions F, G, B_k, \bar{u} . Note that we take the superposition of V^P and $\psi_{\bar{u}}$ to obtain $V^P \circ \psi_{\bar{u}}$ for the original system and hence V^P when we pass to the original system of coordinates. The same remark concerns the function F , but it does not change anything in the computations. We need to solve the system (3.1) to show that T is well defined by (2.27). To this end we need the appropriate estimates that we show in the first part of this section. In the second part the linear system is solved.

3.1 A priori bounds

In this section we show the estimate in $W_p^2 \times W_p^1$ for the solution of the linear system (3.1) in the maximal regularity regime. The first step is the energy estimate. It is given by the following

Lemma 6. *Let (u, w) be a solution to the system (3.1) with given $(F, G, B, \bar{u}) \in V^* \times L_2 \times L_2(\Gamma) \times W_p^2$, where \bar{u} is small enough to assure*

$$V^P + \bar{u}^{(1)} \geq \theta_3 > 0, \quad (3.2)$$

for some $\theta_3 > 0$. Assume that the friction f is large enough on Γ_{in} and the viscosity μ and the friction on Γ_0 satisfy (1.12). Then

$$\|u\|_{W_2^1} + \|w\|_{L_\infty(L_2)} \leq C [\|F\|_{V^*} + \|G\|_{L_2} + \|B\|_{L_2(\Gamma)} + \|w_{in}\|_{L_2(\Gamma_{in})}], \quad (3.3)$$

where

$$V = \{v \in W_2^1(\Omega) : v \cdot n|_\Gamma = 0\} \quad (3.4)$$

and V^* is the dual space of V .

Proof. We start with two basic observations. First, since $V_{x_1}^P = 0$, by (2.7) we have

$$\partial_{z_1}(V^P \circ \psi_{\bar{u}}) = \frac{1}{V^P} [\bar{u}^{(2)} \partial_{x_2} V^P + \bar{u}^{(3)} \partial_{x_3} V^P].$$

Now recall that the constant θ in (1.5) is independent of the smallness of perturbation. Hence we can assume the $\|\bar{u}\|_{W_p^2}$ is small compared to θ and by the imbedding $W_p^1 \subset L_\infty$ we have

$$\|\partial_{z_1}(V^P \circ \psi_{\bar{u}})\|_{W_p^1} \leq E(\bar{u}, V^P). \quad (3.5)$$

We keep in mind that (1.12) and (1.6) hold, what implies that (3.5) will be controlled by the viscosity, more precisely even by a constant which decreases with increasing viscosity due to (1.6). In the remaining of this section we will write V^P instead of $V^P \circ \psi_{\bar{u}}$. The fact that we consider the superposition does not influence the computations as we have (3.5). We apply the identities

$$\int_\Omega V^P \partial_{z_1} |u|^2 dx = \frac{1}{2} \int_\Gamma V^P |u|^2 n^{(1)} d\sigma - \int_\Omega |u|^2 \partial_{z_1} V^P dx \quad (3.6)$$

and

$$\begin{aligned} \int_\Omega (-\mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u) \cdot v dx &= \int_\Omega 2\mu \mathbf{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v dx - \\ &\int_\Gamma n \cdot [2\mu \mathbf{D}(u)] \cdot v d\sigma - \int_\Gamma n \cdot [\nu (\operatorname{div} u) \mathbf{Id}] \cdot v d\sigma. \end{aligned} \quad (3.7)$$

Now we multiply (3.1)₁ by u and integrate. Using the above identities, with application of the boundary conditions (3.1)_{3,4} we arrive at

$$\begin{aligned} \int_\Omega \{2\mu \mathbf{D}(u) : \mathbf{D}(u) + \nu |\operatorname{div} u|^2\} dx &+ \int_{\Gamma_{in}} (f - \frac{V^P}{2}) |u|^2 d\sigma + \int_{\Gamma_{out}} (\frac{f}{2} + V^P) |u|^2 d\sigma \\ &- \gamma \int_\Omega w \operatorname{div} u dx + \int_\Omega (u \cdot \nabla \bar{V}) \cdot u dx = \\ &= \int_\Omega F \cdot u dx + \int_\Gamma \{B_1(u \cdot \tau_1) + B_2(u \cdot \tau_2)\} d\sigma + \int_\Omega |u|^2 \partial_{z_1} V^P dx. \end{aligned} \quad (3.8)$$

Note that by (3.5) we have

$$\left| \int_{\Omega} |u|^2 \partial_{z_1} V^P dx \right| \leq E(\bar{u}, \nabla V^P) \|u\|_{L_2}^2.$$

The other terms on the r.h.s. are all 'good' terms. The Γ_{out} term on the l.h.s. is nonnegative and the Γ_{in} term will be positive for f large enough. To deal with the term $\int_{\Omega} w \operatorname{div} u dx$ we apply the continuity equation to express $\operatorname{div} u$ obtaining:

$$\begin{aligned} \int_{\Omega} w \operatorname{div} u dx &= - \int_{\Omega} G w dx - \frac{1}{2} \int_{\Omega} w^2 \partial_{z_1} (V^P + \bar{u}^{(1)}) dx \\ &+ \frac{1}{2} \int_{\Gamma_{out}} w^2 (V^P + \bar{u}^{(1)}) d\sigma - \frac{1}{2} \int_{\Gamma_{in}} w_{in}^2 (V^P + \bar{u}^{(1)}) d\sigma. \end{aligned}$$

By (3.2) the integral over Γ_{out} will be nonnegative, hence we have

$$\int_{\Omega} w \operatorname{div} u dx \leq \|G\|_{L_2} \|w\|_{L_2} + \|\partial_{z_1} (V^P + \bar{u}^{(1)})\|_{L_{\infty}} \|w\|_{L_2}^2 + C \|w_{in}\|_{L_2(\Gamma_{in})}^2. \quad (3.9)$$

To derive the W_2^1 - norm of u we apply the Korn inequality [23, 27]:

$$\int_{\Omega} [2\mu \mathbf{D}(u) : \mathbf{D}(u) + \nu |\operatorname{div} u|^2] dx + \int_{\Gamma_{in}} f (u \cdot \tau)^2 d\sigma \geq C_K \|u\|_{W_2^1}^2, \quad (3.10)$$

where $C_K = C_K(\mu, \nu, f, \Omega)$ and C_K is increasing with μ . A sketch of the proof of (3.10) one can find in the Appendix, note that in (3.10) we use only information at Γ_{in} , a part of the boundary, but still it is sufficient to control the whole norm of W_2^1 .

Combining (3.8), (3.9) and (3.10) we get

$$\begin{aligned} C_K \|u\|_{W_2^1}^2 &\leq \|G\|_{L_2} \|w\|_{L_2} + C \|w_{in}\|_{L_2(\Gamma_{in})}^2 \\ &+ [\|F\|_{V^*} + \|B\|_{L_2(\Gamma)}] \|u\|_{W_2^1} + E(\bar{u}, \nabla V^P) [\|w\|_{L_2}^2 + \|u\|_{L_2}^2] - \int_{\Omega} (u \cdot \nabla \bar{V}) \cdot u dx. \end{aligned} \quad (3.11)$$

We have to deal with the last term on the r.h.s. It is impossible to show it has a good sign, hence the only way is to estimate it directly with

$$\left| \int_{\Omega} (u \cdot \nabla \bar{V}) \cdot u dx \right| \leq C_P^2 \|\nabla \bar{V}\|_{L_{\infty}} \|u\|_{W_2^1}^2, \quad (3.12)$$

where C_P is the constant from the Poincaré inequality in V . Inserting the above to (3.11) we get

$$\begin{aligned} &(C_K - C_P^2 (\|\nabla \bar{V}\|_{L_{\infty}} + E(\bar{u}, \nabla V^P))) \|u\|_{W_2^1}^2 \leq \\ &\leq E \|w\|_{L_2}^2 + C (\|G\|_{L_2} + \|w_{in}\|_{L_2(\Gamma_{in})}) + (\|F\|_{V^*} + \|B\|_{L_2(\Gamma)}) \|u\|_{W_2^1}, \end{aligned} \quad (3.13)$$

where we recall that E is a small constant and C is a data-dependent constant, not necessarily small. Now, C_K is increasing with μ , while C_P does not depend on μ . Moreover, (1.6) implies that $E(\bar{u}, \nabla V^P)$ will be decreasing when μ increases. Finally, (1.8) implies that for small values

of the friction f on Γ_0 it is enough to assume that the viscosity is large only compared to $f|_{\Gamma_0}$ to control $E(\bar{u}, \nabla V^P)$. We conclude that the constant on the l.h.s. of (3.13) will be positive provided that μ satisfies (1.12).

Here it is a good point to emphasize the necessity of sufficient magnitude of the viscosity coefficient for large f at Γ_0 . This assumption is somehow natural, although in the case $V^P = \text{const}$ it is not required [19]. We have to control (3.12), and largeness of dissipation may only give us this chance. Note that for the Dirichlet boundary condition [9], although the constant flow is considered, such assumption is required, too.

To complete the proof of (3.3) we find a bound on $\|w\|_{L_\infty(L_2)}$. To this end we refer to the next subsection, where we solve the linear system. Notice that $w = S(G - \text{div } u)$, where S is defined in (3.35), and so

$$\|w\|_{L_\infty(L_2)} \leq C (\|G\|_{L_2} + \|u\|_{W_2^1}). \quad (3.14)$$

Combining (3.13) and (3.14) we conclude (3.3). \square

In the next step we show higher bound on the vorticity of the velocity. To this end we take the vorticity of (3.1)₁. Denoting $\alpha = \text{rot } u$ we get

$$\begin{aligned} -\mu \Delta \alpha &= \text{rot} [F - V^P \partial_{z_1} u - u \cdot \nabla \bar{V}] && \text{in } \Omega, \\ \alpha \cdot \tau_2 &= (2\chi_1 - \frac{f}{\nu})u \cdot \tau_1 + \frac{B_1}{\nu} && \text{on } \Gamma, \\ \alpha \cdot \tau_1 &= (\frac{f}{\nu} - 2\chi_2)u \cdot \tau_2 - \frac{B_2}{\nu} && \text{on } \Gamma, \\ \text{div } \alpha &= 0 && \text{on } \Gamma. \end{aligned} \quad (3.15)$$

The boundary conditions (3.15)_{2,3} are derived from differentiation of (3.1)₄ in tangential directions and application of (3.1)₃, see [15], [19]. The above system gives the estimate ([27], Theorem 10.3 with $\mu = 0$):

$$\begin{aligned} \|\alpha\|_{W_p^1} &\leq C [\|F\|_{L_p} + \|\bar{V}\|_{W_\infty^1} \|u\|_{W_p^1} + \|u\|_{W_p^{1-1/p}(\Gamma)} + \|B\|_{W_p^{1-1/p}(\Gamma)}] \\ &\leq C [\|F\|_{L_p} + \|B\|_{W_p^{1-1/p+p}(\Gamma)} + \|u\|_{W_p^1}]. \end{aligned}$$

Applying the interpolation inequality (5.7) to $\|u\|_{W_p^1}$ and the energy estimate we get

$$\|\alpha\|_{W_p^1} \leq C(\epsilon) [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}] + \epsilon \|u\|_{W_p^2} \quad (3.16)$$

for any $\epsilon > 0$. Now consider the Helmholtz decomposition of the velocity

$$u = \nabla \phi + A, \quad (3.17)$$

where $\frac{\partial \phi}{\partial n}|_\Gamma = 0$ and $\text{div } A = 0$. We see that the field A satisfies the system

$$\begin{aligned} \text{rot } A &= \alpha && \text{in } \Omega, \\ \text{div } A &= 0 && \text{in } \Omega, \\ A \cdot n &= 0 && \text{on } \Gamma. \end{aligned} \quad (3.18)$$

This is the classical rot-div system and from [23] we have $\|A\|_{W_p^2} \leq C \|\alpha\|_{W_p^1}$, what by (3.16) can be rewritten as

$$\|A\|_{W_p^2} \leq C(\epsilon) [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}] + \epsilon \|u\|_{W_p^2} \quad (3.19)$$

for any $\epsilon > 0$. Now we substitute the Helmholtz decomposition to (3.1)₁. We get

$$\begin{aligned} & \nabla [-(\nu + 2\mu)\Delta\phi + \gamma w] = \\ & F - V^P \partial_{z_1} A + \mu \Delta A + (\nu + \mu) \nabla \operatorname{div} A - A \cdot \nabla \bar{V} - V^P \partial_{z_1} \nabla \phi - \nabla \phi \cdot \nabla \bar{V} =: \bar{F}. \end{aligned} \quad (3.20)$$

Since $\Delta\phi = \operatorname{div} u$, we can write

$$-(\nu + 2\mu) \operatorname{div} u + \gamma w = \bar{H}. \quad (3.21)$$

We underline that we are now at the level of a priori estimates and (3.21) should be treated as the definition of \bar{H} . In fact we can think of \bar{H} as a kind of effective viscous flux like in the theory of weak solutions to compressible Navier-Stokes equations ([7],[12]). Now (3.20) can be rewritten as $\nabla \bar{H} = \bar{F}$. Combining the last equation with (3.1)₂ we arrive at

$$\bar{\gamma} w + V^P \partial_{z_1} w = H, \quad (3.22)$$

where $\bar{\gamma} = \frac{\gamma}{\nu + 2\mu}$ and

$$H = \frac{\bar{H}}{\nu + 2\mu} + G. \quad (3.23)$$

The equation (3.22) makes it possible to estimate $\|w\|_{W_p^1}$ and $\|\partial_{z_1} w\|_{W_p^1}$ in terms of H . Next we can find the bound on H using interpolation and the energy estimate. The first step is in the following lemma:

Lemma 7. *Let w solve (3.22) with $H \in W_p^1$ and $w|_{\Gamma_{in}} = w_{in}$. Then*

$$\|w\|_{W_p^1} + \|\partial_{z_1} w\|_{W_p^1} \leq C [\|H\|_{W_p^1} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (3.24)$$

Proof. To estimate $\|w\|_{L_p}$ we multiply (3.22) by $|w|^{p-2}w$ and integrate. Using the boundary conditions we get

$$\begin{aligned} & \bar{\gamma} \|w\|_{L_p}^p + \frac{1}{p} \int_{\Gamma_{out}} |w|^p V^P d\sigma \leq \\ & \leq \|H\|_{L_p} \|w\|_{L_p}^{p-1} + \frac{1}{p} \int_{\Gamma_{in}} |w|^p V^P d\sigma + (\|\partial_{z_1} V^P\|_{L_\infty} + \|\bar{u}\|_{W_\infty^1}) \|w\|_{L_p}^{p-1}. \end{aligned}$$

The boundary term on the l.h.s. is positive and the constant in the last term on the r.h.s. is small (note that we take only the z_1 derivative of V^P). Hence the above implies

$$\|w\|_{L_p} \leq C [\|H\|_{L_p} + \|w_{in}\|_{L_p(\Gamma_{in})}]. \quad (3.25)$$

In order to find a bound on $\partial_{z_i} w$ we differentiate (3.22) with respect to z_i . If we assume that $w \in W_p^1$ then (3.22) implies $\partial_{z_1} w \in W_p^1$, since W_p^1 is an algebra. Thus we differentiate (3.22) with respect to z_i , multiply by $|\partial_{z_i} w|^{p-2} \partial_{z_i} w$ and integrate. We have

$$\begin{aligned} \int_{\Omega} (V^P + \bar{u}^{(1)}) |\partial_{z_i} w|^{p-2} \partial_{z_i} w \partial_{z_1} w &= \frac{1}{p} \int_{\Omega} (V^P + \bar{u}^{(1)}) \partial_{z_1} |\partial_{z_i} w|^p dx = \\ &= \int_{\Gamma} (V^P + \bar{u}^{(1)}) |\partial_{z_i} w|^p n^{(1)} d\sigma - \int_{\Omega} |\partial_{z_i} w|^p \partial_{z_1} (V^P + \bar{u}^{(1)}). \end{aligned} \quad (3.26)$$

The last term on the r.h.s. can be estimated by $E \|\partial_{z_i} w\|_{L_p}$ since we take only z_1 derivative of V^P . The boundary term vanishes on Γ_0 and Γ_{out} part will be nonnegative. Hence (3.26) implies

$$\bar{\gamma} \|\partial_{z_i} w\|_{L_p}^p \leq \|H\|_{W_p^1} \|\partial_{z_i} w\|_{L_p}^{p-1} + \frac{1}{p} \int_{\Gamma_{in}} |\partial_{z_i} w_{in}|^p (V^P + \bar{u}^{(1)}) d\sigma + E \|\partial_{z_i} w\|_{L_p}^p. \quad (3.27)$$

For $i = 2, 3$ we use the fact that $w_{in} \in W_p^1(\Gamma_{in})$ and conclude that

$$\|\partial_{z_i} w\|_{L_p} \leq C [\|H\|_{W_p^1} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]. \quad (3.28)$$

To apply this method to $\partial_{z_1} w$ we need some knowledge on $\partial_{z_1} w_{in}|_{\Gamma_{in}}$. To this end we can use (3.22), which, since $V^P + \bar{u}^{(1)} > 0$, can be rewritten on Γ_{in} as

$$\partial_{z_1} w_{in} = \frac{H - \bar{\gamma} w_{in}}{V^P + \bar{u}^{(1)}}.$$

Hence $\|\partial_{z_1} w\|_{L_p(\Gamma_{in})} \leq C[\|H\|_{L_p(\Gamma_{in})} + \|w_{in}\|_{W_p^1(\Gamma_{in})}]$, and (3.27) implies (3.28) also for $i = 1$. From (3.25) and (3.28) we conclude

$$\|w\|_{W_p^1} \leq C [\|H\|_{L_p} + \|w_{in}\|_{L_p(\Gamma_{in})}].$$

The bound on $\|\partial_{x_1} w\|_{W_p^1}$ results simply from the identity (3.22) and the fact that W_p^1 is an algebra. The proof of (3.24) is complete. \square

Now we need to find the bound on $\|H\|_{W_p^1}$, but this is straightforward. Interpolation inequality (5.7) yields

$$\|H\|_{L_p} \leq \delta \|\nabla H\|_{L_p} + C(\delta) \|H\|_{L_2},$$

for any $\delta > 0$. To estimate $\|H\|_{L_2}$ we use the fact that $\bar{H} = -(\nu + 2\mu) \operatorname{div} u + \gamma w$ and the energy estimate (3.3). To find the bound on $\|\nabla H\|_{L_p}$ we use (3.19), (3.20), then (5.7) to estimate the term $\|u\|_{W_p^1}$ and finally (3.3). We obtain

$$\|H\|_{W_p^1} \leq C(\delta) [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{W_p^1(\Gamma_{in})}] + \delta \|u\|_{W_p^2}. \quad (3.29)$$

We are now one short step from the main result of this section. It is given by the following

Theorem 2. *Let (u, w) be a solution to (3.1) with $(F, G, B_k, w_{in}, \bar{u}) \in L_p \times W_p^1 \times W_p^{1-1/p}(\Gamma) \times W_p^1(\Gamma_{in}) \times W_p^2$ such that $\|\bar{u}\|_{W_p^2}$ is small enough and the friction f is large enough on Γ_{in} . Assume also that the viscosity and the friction on Γ_0 satisfy (1.12). Then*

$$\|u\|_{W_p^2} + \|w\|_{W_p^1} + \|\partial_{z_1} w\|_{W_p^1} \leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|B_k\|_{W_p^{1-1/p}(\Gamma_{in})} + \|w\|_{W_p^1(\Gamma_{in})}]. \quad (3.30)$$

Proof. To close the estimate (3.30) it remains to find the bound on $\|u\|_{W_p^2}$. To this end notice that in particular u satisfies the Lamé system:

$$\begin{aligned} -\mu\Delta u - (\nu + \mu)\nabla\operatorname{div} u &= F - \gamma\nabla w - V^P\partial_{z_1}u - u \cdot \nabla\bar{V} & \text{in } \Omega, \\ n \cdot 2\mu\mathbf{D}(u) \cdot \tau_i + f \, u \cdot \tau_i &= B_i, \quad i = 1, 2 & \text{on } \Gamma, \\ n \cdot u &= 0 & \text{on } \Gamma. \end{aligned} \quad (3.31)$$

Lemma 11 applied to the above system yields

$$\|u\|_{W_p^2} \leq C [\|F\|_{L_p} + \|w\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|u\|_{W_p^1}].$$

Applying the interpolation inequality to the term $\|u\|_{W_p^1}$ and then the energy estimate (3.3) we get

$$\|u\|_{W_p^2} \leq C [\|F\|_{L_p} + \|G\|_{W_p^1} + \|w\|_{W_p^1} + \|B\|_{W_p^{1-1/p}(\Gamma)} + \|w_{in}\|_{L_2(\Gamma_{in})}]. \quad (3.32)$$

Combining this estimate with (3.24) and (3.29) with appropriate δ we conclude (3.30). \square

Note that Theorem 2 gives more than we need to solve (3.1), namely the bound in W_p^1 of $\partial_{z_1}w$. We will use this result to show the contraction property for the operator T in Section 4.

3.2 Solution of the linear system

With the estimates that we obtained we are ready to solve the system (3.1). First we define the weak solution and show its existence. Next, applying the estimate (3.30) we show its regularity under the appropriate regularity of the data.

3.2.1 Weak solution

By the weak solution to (3.1) we mean a couple $(u, w) \in V \times L_\infty(L_2)$ such that

$$\begin{aligned} \int_{\Omega} \{v \cdot V^P\partial_{z_1}u + u \cdot \nabla\bar{V}\} + 2\mu\mathbf{D}(u) : \nabla v + \nu \operatorname{div} u \operatorname{div} v - \gamma w \operatorname{div} v \} dx \\ + \int_{\Gamma} f(u \cdot \tau_i)(v \cdot \tau_i) d\sigma = \int_{\Omega} F \cdot v dx + \int_{\Gamma} B_i(v \cdot \tau_i) d\sigma \end{aligned} \quad (3.33)$$

is satisfied $\forall v \in V$ and (3.1)₂ is satisfied in $\mathcal{D}'(\Omega)$, i.e. for all $\phi \in \bar{C}^\infty(\Omega)$, $\phi|_{\Gamma_{out}} = 0$:

$$-\int_{\Omega} (V^P + \bar{u}^{(1)})w\partial_{z_1}\phi dx - \int_{\Omega} \partial_{z_1}(V^P + \bar{u}^{(1)})w\phi dx = \int_{\Omega} \phi(G - \operatorname{div} u) dx + \int_{\Gamma_{in}} (V^P + \bar{u}^{(1)})w_{in}\phi d\sigma. \quad (3.34)$$

To find the weak solution we apply the Galerkin method. Hence we introduce an orthonormal basis of $\omega_k \subset V$ and finite dimensional spaces $V^N = \{\sum_{i=1}^N \alpha_i \omega_i : \alpha_i \in \mathbb{R}\} \subset V$. We look

for the approximations of the velocity of the form $u^N = \sum_{i=1}^N c_i^N \omega_i$. Taking into account the

continuity equation we have to define the approximations of the density in an appropriate way. Namely, we set $w^N = S(G^N - \operatorname{div} u^N)$, where $S : L_2(\Omega) \rightarrow L_\infty(L_2)$ is defined as

$$w = S(v) \iff \begin{cases} (V^P + \bar{u}^{(1)}) \partial_{z_1} w = v & \text{in } \mathcal{D}'(\Omega), \\ w = w_{in} & \text{on } \Gamma_{in} \end{cases} \quad (3.35)$$

and satisfies the estimate

$$\|S(v)\|_{L_\infty(L_2)} \leq C [\|w_{in}\|_{L_2(\Gamma_{in})} + \|v\|_{L_2}]. \quad (3.36)$$

The construction of S is quite straightforward. For a continuous v we set

$$S(v)(z) = w_{in}(0, z_2, z_3) + \int_0^{z_1} \frac{v}{V^P + \bar{u}^{(1)}}(s, z_2, z_3) ds. \quad (3.37)$$

Next we show directly the estimate (3.36), which enables us to extend S on $L_2(\Omega)$ using a standard density argument.

Now we proceed with the Galerkin scheme. Taking $F = F^N$, $u = u^N = \sum_i c_i^N \omega_i$, $v = \omega_k$, $k = 1 \dots N$ and $w = w^N = S(G^N - \operatorname{div} u^N)$ in (3.33), where F^N and G^N are orthogonal projections of F and G on V^N , we arrive at a system of N equations

$$B^N(u^N, \omega_k) = 0, \quad k = 1 \dots N, \quad (3.38)$$

where $B^N : V^N \rightarrow V^N$ is defined as

$$\begin{aligned} B^N(\xi^N, v^N) = & \int_\Omega \{v^N V^P \partial_{z_1} \xi^N + \xi^N \cdot \nabla \bar{V} + 2\mu \mathbf{D}(\xi^N) : \nabla v^N + \operatorname{div} \xi^N \operatorname{div} v^N\} dx \\ & - \gamma \int_\Omega S(G^N - \operatorname{div} \xi^N) \operatorname{div} v^N dx + \int_\Gamma [f(\xi^N \cdot \tau_j) - B_i] (v^N \cdot \tau_j) d\sigma - \int_\Omega F^N \cdot v^N dx. \end{aligned} \quad (3.39)$$

Now, if u^N satisfies (3.38) for $k = 1 \dots N$ and $w^N = S(G^N - \operatorname{div} u^N)$, then a pair (u^N, w^N) satisfies (3.33) - (3.34) for $(v, \phi) \in (V^N \times \bar{C}^\infty(\Omega))$, $\phi|_{\Gamma_{out}} = 0$. We will call such a pair an approximate solution to (3.33) - (3.34). To solve the system (3.38) we apply the following well-known result (the proof can be found in [25]):

Lemma 8. *Let X be a finite dimensional Hilbert space and let $P : X \rightarrow X$ be a continuous operator satisfying*

$$\exists M > 0 : \quad (P(\xi), \xi) > 0 \quad \text{for } \|\xi\| = M. \quad (3.40)$$

Then $\exists \xi^ : \quad \|\xi^*\| \leq M \quad \text{and} \quad P(\xi^*) = 0$.*

We define $P^N : V^N \rightarrow V^N$ as

$$P^N(\xi^N) = \sum_k B^N(\xi^N, \omega_k) \omega_k \quad \text{for } \xi^N \in V^N. \quad (3.41)$$

In order to apply Lemma 8 we show that $(P(\xi^N), \xi^N) > 0$ on some sphere in V^N with radius dependent on the norms of the data. To this end we follow the proof of the energy estimate for (3.1). This is in fact standard approach in the Galerkin method: the energy estimate combined

with Lemma 8 gives existence of the approximate solutions, hence we skip the details here. Except from the existence of the approximate solution u^N , Lemma 8 gives the estimate

$$\|u^N\|_{W_2^1} \leq C(DATA),$$

which combined with (3.36) gives

$$\|u^N\|_{W_2^1} + \|w^N\|_{L_\infty(L_2)} \leq C(DATA).$$

Thus

$$u^N \rightharpoonup u \text{ in } W_2^1 \quad \text{and} \quad w^N \rightharpoonup^* w \text{ in } L_\infty(L_2)$$

for some $(u, w) \in W_2^1 \times L_\infty(L_2)$. We easily to verify that (u, w) is a weak solution. First, passing to the limit in (3.33) for (u^N, w^N) we see that u satisfies (3.33) with w . On the other hand, taking the limit in (3.34) we verify that $w = S(G - \operatorname{div} u)$. We conclude that (u, w) satisfies (3.33) - (3.34), thus we have the weak solution. To show the boundary condition on the density we can rewrite the r.h.s of (3.35) as

$$\begin{cases} \partial_{z_1} w = \frac{v}{V^P + \bar{u}(1)} & \text{in } \mathcal{D}'(\Omega), \\ w = w_{in} & \text{on } \Gamma_{in}, \end{cases} \quad (3.42)$$

and, treating x_1 as a 'time' variable, adapt Di Perna - Lions theory of transport equation ([6]) that implies the uniqueness of solution to (3.42) in the class $L_\infty(L_2)$ (note that this is the reason we work with weak solutions with the density in $L_\infty(L_2)$ instead of usual L_2). This completes the proof of existence of weak solution.

3.2.2 Strong solution

The following result gives strong solution to the linear system (3.1) for the data of appropriate regularity.

Theorem 3. *Let $(F, G, B_k, \bar{u}) \in (L_p \times W_p^1 \times W_p^{1-1/p}(\Gamma) \times W_p^2)$ with $\|\bar{u}\|_{W_p^2}$ small enough. Assume further that f is large enough on Γ_{in} and μ fulfills (1.12). Then there exist a unique solution $(u, w) \in W_p^2 \times W_p^1$ to the system (3.1) and the estimate (3.30) holds.*

Proof. To show appropriate regularity of the weak solution for the regular data it is enough to apply the estimate (3.30) provided that we handle the singularities of the boundary at the junctions of the wall Γ_0 with inlet Γ_{in} and outlet Γ_{out} . To this end we apply the result on the elliptic regularity of the Lamé system with slip boundary conditions, Lemma 11 in the Appendix. Notice that we can apply this method since we work in the fixed domain Ω due to the appropriate choice of the change of variables. Otherwise we would end up in a free boundary problem and the solution of the linear system would be highly nontrivial. \square

4 Contraction

In this section we show the contraction property for the operator T defined in (2.27). However, first of all we notice that Theorem 3 gives the solution of the linear system (3.1) provided that $\|\bar{u}\|_{W_p^2}$ is small enough, and so this constraint must hold if we want to have $T(\bar{u}, \bar{w})$ well defined. We start this section with showing a stronger result, namely that $T : B_R \rightarrow B_R$ for some ball $B_R \in W_p^2 \times W_p^1$ with R depending on the data. This property will also be needed to show the contraction. Eventually, we prove Theorem 1, the main result of the paper.

Lemma 9. *There exists $R > 0$ depending on the size of the data measured by D_0 (1.11) such that, provided the data is small enough, $T(B_R) \subset B_R$, where B_R is a ball of radius R in $W_p^2 \times W_p^1$.*

Proof. The estimates (2.21) and (3.30) imply that

$$\|T(\bar{u}, \bar{w})\|_{W_p^2 \times W_p^1} \leq C \|(\bar{u}, \bar{w})\|_{W_p^2 \times W_p^1}^2 + \delta,$$

where $\delta < \frac{1}{4C}$, provided that the data is small enough. For such δ we have

$$\|(\bar{u}, \bar{w})\|_{W_p^2 \times W_p^1} < 2\delta \Rightarrow \|T(\bar{u}, \bar{w})\|_{W_p^2 \times W_p^1} < 2\delta. \quad \square$$

Now we show the contraction property for T . To this end consider $(u_1, w_1) = T(\bar{u}_1, \bar{w}_1)$, $(u_2, w_2) = T(\bar{u}_2, \bar{w}_2)$. By the definition of T , the difference $(u, w) := (u_1 - u_2, w_1 - w_2)$ satisfies the system

$$\begin{aligned} u \cdot \nabla_z \bar{V} + V^P \partial_{z_1} u - \mu \Delta_z u - (\mu + \nu) \nabla_z \operatorname{div}_z u + \gamma \nabla_z w &= \tilde{F}(\bar{u}_1, \bar{w}_1) - \tilde{F}(\bar{u}_2, \bar{w}_2), \\ (V^P + \bar{u}_2^{(1)}) \partial_{z_1} w + \operatorname{div}_z u &= \tilde{G}(\bar{u}_1, \bar{w}_1) - \tilde{G}(\bar{u}_2, \bar{w}_2) + (\bar{u}_1 - \bar{u}_2) \partial_{z_1} w_1, \\ n \cdot 2\mu \mathbf{D}_z(u) \cdot \tau_k + f(u \cdot \tau_k)|_\Gamma &= n \cdot 2\mu [R(u_2, \mathbf{D}) - R(u_1, \mathbf{D})] \cdot \tau_k, \\ n \cdot u|_\Gamma &= 0, \quad w|_{\Gamma_{in}} = 0. \end{aligned} \tag{4.1}$$

Hence in order to show the contraction principle for T we can apply (3.30) provided that we have good bounds on the r.h.s. of (4.1). A result we need is given by the following

Lemma 10. *We have*

$$\begin{aligned} &\|\tilde{F}(\bar{u}_1, \bar{w}_1) - \tilde{F}(\bar{u}_2, \bar{w}_2)\|_{L_p} + \|\tilde{G}(\bar{u}_1, \bar{w}_1) - \tilde{G}(\bar{u}_2, \bar{w}_2)\|_{W_p^1} \\ &+ \|(\bar{u}_1 - \bar{u}_2) \partial_{z_1} w_1\|_{W_p^1} + \|n \cdot 2\mu [R(u_2, \mathbf{D}) - R(u_1, \mathbf{D})] \cdot \tau_k\|_{W_p^{1-1/p}(\Gamma)} \\ &\leq E [\|u_1 - u_2\|_{W_p^2} + \|w_1 - w_2\|_{W_p^1}]. \end{aligned} \tag{4.2}$$

Proof. Denote (\bar{u}_i, \bar{w}_i) by (u_i, w_i) for simplicity. We have

$$\begin{aligned} \tilde{F}(u_1, w_1) - \tilde{F}(u_2, w_2) &= u_1 \cdot \nabla_x u_1 - u_2 \cdot \nabla_x u_2 + (u_1 - u_2) \cdot \nabla_x u_0 + u_0 \cdot \nabla_x (u_1 - u_2) \\ &\quad + \delta \pi'(w_1) \nabla_x w_1 - \delta \pi'(w_2) \nabla_x w_2 + (w_1 - w_2) F + \tilde{R}, \end{aligned}$$

where

$$\delta \pi'(w_i) := \pi'(w_i + 1) - \pi'(w_i)$$

and \tilde{R} denotes all the differences between corresponding commutators in \tilde{F} . We can afford using such abbreviation and estimate \tilde{R} without any additional computation if we just notice that the commutators are linear with respect to the functions and hence if certain estimate in terms of the function holds for a commutator, then the same estimate in terms of the difference hold for the difference of the commutators, for example

$$|R(u, \partial_{x_1})| \leq E \|u\|_{W_p^1} \Rightarrow |R(u_1, \partial_{x_1}) - R(u_2, \partial_{x_1})| \leq E \|u_1 - u_2\|_{W_p^1}.$$

Applying this reasoning to all the commutators in \tilde{F} we conclude that

$$\|\tilde{R}\|_{L_p} \leq E(\phi) (\|u_1 - u_2\|_{W_p^2} + \|w_1 - w_2\|_{W_p^1}). \quad (4.3)$$

We estimate the remaining parts. Obviously we have

$$\|(u_1 - u_2) \cdot \nabla_x u_0\|_{L_p} + \|u_0 \cdot \nabla_x (u_1 - u_2)\|_{L_p} \leq E(u_0, \phi) \|u_1 - u_2\|_{W_p^2}, \quad (4.4)$$

where $E(\cdot, \cdot)$ depends also on ϕ since we have commutators as the gradients are w.r.t. x . A little bit closer examination shows that the i -th coordinate

$$\begin{aligned} (u_1 \cdot \nabla_x u_1 - u_2 \cdot \nabla_x u_2)^{(i)} &= (u_1 \cdot \nabla_x (u_1 - u_2) + (u_1 - u_2) \cdot \nabla_x u_2)^{(i)} = \\ &= \sum_j [(u_1 - u_2)^{(j)} \partial_{z_j} u_2^{(i)} + R(u_2^{(i)}, \partial_{x_j})] + \sum_j [u_1^{(j)} \partial_{z_j} (u_1 - u_2)^{(i)} + R((u_1 - u_2)^{(i)}, \partial_{x_j})], \end{aligned}$$

and so by a direct computation we get

$$\|u_1 \cdot \nabla_x u_1 - u_2 \cdot \nabla_x u_2\|_{L_p} \leq E(\|u_i\|_{W_p^2}, \phi) \|u_1 - u_2\|_{W_p^2}. \quad (4.5)$$

It remains to estimate

$$\begin{aligned} \delta\pi'(w_1) \nabla_x w_1 - \delta\pi'(w_2) \nabla_x w_2 &= \delta\pi'(w_1) \nabla_x (w_1 - w_2) + [\delta\pi'(w_1) - \delta\pi'(w_2)] \nabla_x w_2 = \\ &= \delta\pi'(w_1) [\nabla_z (w_1 - w_2) + R(w_1 - w_2, \nabla)] + [\delta\pi'(w_1) - \delta\pi'(w_2)] [\nabla_z w_2 + R(w_2, \nabla)]. \end{aligned}$$

It follows easily that

$$\|\delta\pi'(w_1) \nabla_x w_1 - \delta\pi'(w_2) \nabla_x w_2\|_{L_p} \leq E(\|w_1\|_{W_p^1}, \|w_2\|_{W_p^1}, \phi) \|w_1 - w_2\|_{W_p^1}. \quad (4.6)$$

Combining (4.3), (4.4), (4.5) and (4.6) we conclude

$$\|\tilde{F}(u_1, w_1) - \tilde{F}(u_2, w_2)\|_{L_p} \leq E(\|w_i\|_{W_p^1}, \|u_i\|_{W_p^2}, \phi) [\|u_1 - u_2\|_{W_p^2} + \|w_1 - w_2\|_{W_p^1}]. \quad (4.7)$$

Now we estimate the difference in G . We have

$$\begin{aligned} \tilde{G}(u_1, w_1) - \tilde{G}(u_2, w_2) &= \\ &= (w_2 - w_1) \operatorname{div} u_0 + w_1 \operatorname{div} (u_1 - u_2) + \operatorname{div} u_2 (w_1 - w_2) + R(u_2, \operatorname{div}) - R(u_1, \operatorname{div}). \end{aligned}$$

The first term

$$\begin{aligned} \|(w_2 - w_1) \operatorname{div} u_0\|_{W_p^1} &\leq \|\nabla(w_2 - w_1) \operatorname{div} u_0\|_{L_p} + \|(w_2 - w_1) \nabla^2 u_0\|_{L_p} \leq \\ &\leq \|\operatorname{div} u_0\|_{L_\infty} \|w_1 - w_2\|_{W_p^1} + \|u_0\|_{W_p^2} \|w_1 - w_2\|_{L_\infty} \leq C \|u_0\|_{W_p^2} \|w_1 - w_2\|_{W_p^1}. \end{aligned}$$

The second

$$\|w_1 \operatorname{div}(u_1 - u_2)\|_{W_p^1} \leq \|(\nabla w_1) \operatorname{div}(u_1 - u_2)\|_{L_p} + \|w_1 \nabla^2(u_1 - u_2)\|_{L_p} \leq C \|w_1\|_{W_p^1} \|u_1 - u_2\|_{W_p^2}.$$

Similarly we show

$$\|\operatorname{div} u_2(w_1 - w_2)\|_{W_p^1} \leq C \|u_2\|_{W_p^2} \|w_1 - w_2\|_{W_p^1}.$$

Now we have to estimate the difference of the commutators in W_p^1 . It turns out to be straightforward as we have

$$R(u_1, \operatorname{div}) - R(u_2, \operatorname{div}) = \sum_i \partial_{z_i}(u_1 - u_2)^{(i)} \phi_{x_i}^{(i)} + \sum_{i \neq j} \partial_{z_j}(u_1^{(1)} - u_2^{(1)})^{(1)} \phi_{x_i}^{(j)}$$

and so, identically as in (2.25), we show that

$$\|R(u_1, \operatorname{div}) - R(u_2, \operatorname{div})\|_{W_p^1} \leq E(\phi) \|(u_1 - u_2)\|_{W_p^2}.$$

Combining the above results we conclude

$$\|\tilde{G}(u_1, w_1) - \tilde{G}(u_2, w_2)\|_{W_p^1} \leq [E(\phi) + C(\|w_1\|_{W_p^1} + \|u_2\|_{W_p^2})](\|u_1 - u_2\|_{W_p^2} + \|w_1 - w_2\|_{W_p^1}). \quad (4.8)$$

To treat the last term of the r.h.s. of (4.1)₂ we observe that

$$\|(\bar{u}_1 - \bar{u}_2) \partial_{z_1} w_1\|_{W_p^1} \leq C \|\partial_{z_1} w_1\|_{W_p^1} \|\bar{u}_1 - \bar{u}_2\|_{W_p^2}. \quad (4.9)$$

It remains to estimate the boundary terms $n \cdot [R(u_1, \mathbf{D}) - R(u_2, \mathbf{D})] \cdot \tau_k$. To this end it is enough to notice that

$$\{R(u_1, \mathbf{D}) - R(u_2, \mathbf{D})\}_{i,j} = R((u_1 - u_2)^{(i)}, \partial_{x_j}) - R((u_1 - u_2)^{(j)}, \partial_{x_i}) =: R_{i,j}.$$

Applying the trace theorem and repeating the proof of (4.8) we can show that

$$\|R_{i,j}\|_{W_p^{1-1/p}(\Gamma)} \leq C \|R_{i,j}\|_{W_p^1} \leq E(\phi) \|u_1 - u_2\|_{W_p^2},$$

and so

$$\|n \cdot [R(u_1, \mathbf{D}) - R(u_2, \mathbf{D})] \cdot \tau_k\|_{W_p^{1-1/p}(\Gamma)} \leq E(\phi) \|u_1 - u_2\|_{W_p^2}. \quad (4.10)$$

Combining (4.7), (4.8), (4.9) and (4.10) we conclude (4.2). \square

Proof of Theorem 1. With the results of the previous section we can apply the Banach fixed point theorem to the operator T . It gives existence of a unique fixed point within the ball $B(0, R) \in W_p^2 \times W_p^1$. By Lemma 9 we have $R = R(D_0)$ where D_0 is defined in (1.11). By the definition of T , the fixed point (u, w) solves the system (2.11).

Now we recall Section 2 and conclude that the original coordinate system is $x = \psi_u(z)$, and in the x variable our solution satisfies the system (2.5). It follows that $v = \bar{v} + u + u_0$ and $\rho = w + 1$ solves (1.1) and the estimate (1.13) holds. Theorem 1 is proved.

5 Appendix

Proof of Lemma 1. As explained in Section 1, we assume the pressure of the form $\bar{\Pi} = \omega(f) x_1$. Then on each x_1 - cut of Ω (i.e. on each set $\Omega_0 \times \{x_1\}$) V^P can be found as a solution to the elliptic problem (1.7) which we recall here:

$$\begin{aligned} \mu \Delta v &= \bar{\Pi}_{x_1} = \omega(f) < 0 \quad \text{in } \Omega_0, \\ \mu \frac{\partial v}{\partial n} + f v &= 0 \quad \text{on } \partial\Omega_0. \end{aligned} \quad (5.1)$$

Testing (1.7) with $v_- = v \chi_{v < 0}$ we get

$$- \int_{\Omega} \mu |\nabla v_-|^2 - \int_{\Omega} f v_-^2 = \int_{\Omega} \omega(f) v_- \geq 0. \quad (5.2)$$

The last inequality results from nonpositivity of v_- and the direction of the flow which implies $\omega(f) < 0$ (recall the remark after (1.7)). Clearly (5.2) implies $v \geq 0$. We want to show sharp inequality. To this end consider \bar{v} satisfying

$$\Delta \bar{v} = \omega(f)/\mu < 0 \quad \text{in } \Omega, \quad \bar{v}|_{\Gamma} = 0 \quad \text{at } \partial\Omega.$$

We have $\bar{v} \geq 0$, and, by the maximum principle applied to $v - \bar{v}$ we get $\inf_{\Omega} v = \inf_{\Gamma} v$. Assume that $\inf_{\Gamma} v = v(x_0) = 0$ for some $x_0 \in \Gamma$. Then, since $f \geq 0$ and $\frac{\partial \bar{v}}{\partial n} \leq 0$, we must have

$$\frac{\partial(v - \bar{v})}{\partial n}(x_0) = -\frac{\partial \bar{v}}{\partial n}(x_0) - f v(x_0) \geq 0.$$

But since $v(x_0) = \inf_{\Omega}(v - \bar{v})$, by the Hopf Lemma we must have $\frac{\partial(v - \bar{v})}{\partial n}(x_0) < 0$. The application of Hopf lemma is possible since Ω_0 is a C^2 subset of \mathbb{R}^2 , hence v is a classical solution to (5.1), i.e. $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$. We conclude that $v \geq \theta > 0$ on Γ , hence $v \geq \theta$ in $\bar{\Omega}$. In particular, the linearity of (5.1) gives (1.6) where $\bar{\omega}$ is continuous since ω is continuous (recall the discussion after the formulation of Lemma 1). This completes the proof. \square

Lemma 11. (*Lamé system with slip boundary conditions*). Let $\mu > 0$, $\nu + 2\mu > 0$, $F \in L_p(\Omega)$ and $B \in W_p^{1-1/p}(\Gamma)$. Then there exists $u \in W_p^2(\Omega)$ solving

$$\begin{aligned} -\mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u &= F && \text{in } \Omega, \\ n \cdot 2\mu \mathbf{D}(u) \cdot \tau + f u \cdot \tau &= B, \quad i = 1, 2 && \text{on } \Gamma, \\ n \cdot u &= 0 && \text{on } \Gamma. \end{aligned} \quad (5.3)$$

Moreover, the following estimate holds:

$$\|u\|_{W_p^2} \leq C[\|F\|_{L_p} + \|B\|_{W_p^{1-1/p}(\Gamma)}]. \quad (5.4)$$

Proof. Under the assumptions on μ and ν (5.3) is elliptic so we easily get a weak solution. The only problem we encounter showing regularity of the weak solution under appropriate regularity of the data are the singularities of the boundary on the junctions of Γ_0 with Γ_{in} and Γ_{out} . These

can be dealt with using symmetry arguments. But first we have to reformulate slightly the system (5.3). Having the weak solution, and the bound in W_2^1 for the velocity, we are able to consider only the case $f \equiv 0$ and $B \equiv 0$. The general case will be a consequence of this particular one. Assume that $\Gamma_{in} \subset \{x_1 = 0\}$, then we define the operator E_{as}^v extending a vector field defined for $\{x : x_1 \geq 0\}$ on the whole space as

$$E_{as}^v(u)(x) = \begin{cases} u(x), & x_1 \geq 0, \\ \tilde{u}(\tilde{x}), & x_1 < 0, \end{cases} \quad (5.5)$$

where $\tilde{x} = (-x_1, x_2, x_3)$ and $\tilde{u}(\tilde{x}) = [-u^{(1)}(x), u^{(2)}(x), u^{(3)}(x)]$. Then we have

$$\Delta E_{as}^v(v) + \nabla \operatorname{div} E_{as}^v(v) = \Delta v + \nabla \operatorname{div} v, \quad (5.6)$$

and on the plane $x_1 = 0$ the extension E_{as}^v preserves the slip boundary conditions for $f \equiv 0$ and $B \equiv 0$, since

$$n \cdot \mathbf{D}(E_{as}^v(v)) \cdot \tau_i|_{x_1=0} = \partial_{\tau_i} E_{as}^v(v)|_+ \partial_{x_1} E_{as}^v(v)^{(i)} = 0$$

and $n \cdot E_{as}^v(v)|_{x_1=0} = 0$, $\partial_{x_1} E_{as}^v(v)^{(i)} = 0$ for $i = 2, 3$. Now we are allowed to localize the equations in a vicinity of Γ_{in} obtaining a system in a smooth domain. Then the theory [2, 3] gives the full regularity of $E_{as}^v(v)$ in neighborhood of the junctions. Application of analogous antisymmetric extension on Γ_{out} completes the proof. \square

Lemma 12. (*interpolation inequality*):

For $f \in W_p^1(\Omega)$, $p > 3$:

$$\|f\|_{L_p} \leq \epsilon \|\nabla f\|_{L_p} + C(\epsilon, p, \Omega) \|f\|_{L_2}. \quad (5.7)$$

Proof. The interpolation inequality in L_p ([1], Theorem 2.11) and the imbedding $W_p^1 \subset L_\infty$ for $p > 3$ yields

$$\|f\|_{L_p} \leq C(p) \|f\|_{L_\infty}^\theta \|f\|_{L_2}^{1-\theta} \leq C(p) (\|f\|_{L_p} + \|\nabla f\|_{L_p})^\theta \|f\|_{L_2}^{1-\theta},$$

what entails (5.7) after application of the Cauchy inequality. \square

Lemma 13. *Let Ω be bounded with sufficiently smooth boundary and Γ_{part} be an open regular subset of $\partial\Omega$. Then*

$$\|u\|_{W_2^1} \leq C(\|\mathbf{D}(u)\|_{L_2} + \|u|_{\Gamma_{part}}\|_{L_2(\Gamma_{part})}) \quad (5.8)$$

for $u \in W_2^1(\Omega)$.

Proof. The known result based on properties of the kernel of $\mathbf{D}(\cdot)$ [11, 28] yields

$$\|u\|_{W_2^1} \leq C_1 \|\mathbf{D}(u)\|_{L_2} + C_2 \|u\|_{L_2}. \quad (5.9)$$

In order to obtain (5.8) we shall prove that

$$\|u\|_{L_2} \leq \frac{1}{2C_2} \|\nabla u\|_{L_2} + M(\|\mathbf{D}(u)\|_{L_2} + \|u|_{\Gamma_{part}}\|_{L_2(\Gamma_{part})}). \quad (5.10)$$

Compactness argument and features of \mathbf{D} implies (5.10). We use that fact that the only solution to the system $\mathbf{D}(u^*) = 0, u|_{\Gamma_{part}} = 0$ is $u^* \equiv 0$. To show it we use the fact that if $\mathbf{D}(u) = 0$ then

$$u = A \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} + B \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} + C \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}. \quad (5.11)$$

Hence u is an affine map and it is easy to verify that if at least one of the coefficients A, B, C is nonzero then the rank of matrix of u is 2, hence $\text{Ker } u$ is a line. On the other hand, Γ_{part} is a two dimensional submanifold as it is an open, regular subset of $\partial\Omega$. But $\Gamma_{part} \subset \text{Ker } u$, hence we conclude that $A = B = C = 0$, hence $d_i = 0$. \square

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